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NON-HOLONOMIC CHAPLYGIN SYSTEMS*

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The tools developed in /1/ to describe holonomic systems on a tangent bundle, and some concepts introduced in /2/, are applied to non-holonomic Chaplygin systems. Through the use of differential forms, the concept of quasicoordinates is defined, and the conditions studied in /3/ for the existence of a Chaplygin reducing factor for systems with an arbitrary number of degrees of freedom are formulated. The integral invariant is shown to have properties characteristic for Chaplygin systems and different from those established in /4/. The existence of an invariant measure with density differentiable with respect to velocities in a potential-free system implies the existence of a measure with density dependent only on the coordinates. The invariance of a certain measure with density dependent on the coordinates (but not on the velocities) in a potential-free system implies invariance of the same measure after a potential has been added. As an example it is proved that the equations of motion for a non-holonomic Chaplygin sphere /5/ with arbitrary potential admit of an invariant measure.

1. Statement of the problem. Quasicoordinates. Let TV^n be the tangent bundle of a configuration manifold V^n with local coordinates $(\mathbf{q}, \mathbf{q}^{\cdot})$. We define an operator d_v acting on functions by

$$d_v f = \sum_{i=1}^{n} \frac{\partial f}{\partial q_i^{\cdot}} dq_i \tag{1.1}$$

Generalized forces will be represented by a 1-form

$$Q = \sum_{i=1}^{n} Q_i(\mathbf{q}, \mathbf{q}^{\cdot}) dq_i$$

If T is the kinetic energy, $\det \| T_{q^{\cdot}q^{\cdot}} \| \neq 0$, then the equation

$$dd_vT(\mathbf{X}, \cdot) = -dT + Q \tag{1.2}$$

defines a vector field $\mathbf{X}$ corresponding to the mechanical system /1/. Here dd_vT is a closed non-singular differential 2-form, and dT, Q are differential 1-forms.

The form dd_vT has a non-singular skew-symmetric matrix, and therefore the components of the field $\mathbf{X}$ are determined as solutions of a system of linear equations; it has been shown /2/ that the field $\mathbf{X}$ is "special", i.e.,

$$\mathbf{X} = \sum_{i=1}^{n} \left[a_i \frac{\partial}{\partial q_i^{\cdot}} + q_i^{\cdot} \frac{\partial}{\partial q_i} \right] \tag{1.3}$$

We consider a vector field defined in a more general way than (1.2), by the equation

$$\Omega(\mathbf{X}, \cdot) = R \tag{1.4}$$

where Ω is a (not necessarily closed) 2-form. Let $\mathbf{X}$ be a special field, i.e., satisfying (1.3).

Considering functions defined in a neighbourhood of the set $A = \{q_i^{\cdot} = 0,\ i = 1, \ldots, n\}$, we can define the notion of homogeneity with respect to velocities. This notion can be extended to vector fields and differential forms.

Definition 1. The forms $dq_i^{\cdot}$ and fields $\partial/\partial q_i^{\cdot}$ are homogeneous with respect to velocities

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with degrees of homogeneity one and minus one, respectively. The degree of homogeneity of a product of these objects by homogeneous functions is the sum of the degrees of homogeneity of the factors.

Example. The differential form

$$\Sigma a_{ij}(\mathbf{q})\, dq_i^{\cdot} \wedge dq_j + \Sigma b_{ij}^{k}(\mathbf{q})\, q_k^{\cdot} dq_i \wedge dq_j$$

and vector field

$$\sum c_{ij}^{k}(\mathbf{q})\, q_i^{\cdot} q_j^{\cdot} \frac{\partial}{\partial q_k^{\cdot}} + \sum q_i^{\cdot} \frac{\partial}{\partial q_i}$$

are homogeneous of degree one in the velocities.

Note that the form Ω in Eq.(1.4) is defined up to terms Λ such that

$$\Lambda(\mathbf{X}, \cdot) = 0 \tag{1.5}$$

The coefficients of a 2-form satisfying this condition may be analytic in the velocities, and the form itself has a series expansion in the neighbourhood of the set A, whose terms are homogeneous with respect to the velocities. For example,

$$\Lambda = q_1^{\cdot} dq_2 \wedge dq_3 + q_3^{\cdot} dq_1 \wedge dq_2 + q_2^{\cdot} dq_3 \wedge dq_1$$

However, a form Λ satisfying condition (1.5) may also be constructed by other means.

Definition 2. Let $\theta = \Sigma\theta_i dq_i'$ be a non-singular differential 1-form on V^n, considered as a form on TV^n. A quasicoordinate corresponding to θ is a function π, not necessarily defined throughout the phase space TV^n, which satisfies the equation

$$L_{\mathbf{X}}\pi = \theta(\mathbf{X}) \tag{1.6}$$

throughout its domain of definition.

Remarks. 1°. The right-hand side of the partial differential Eq.(1.6) is linear with respect to the velocities: $\theta(\mathbf{X}) = \Sigma\theta_i(\mathbf{q})\, q_i^{\cdot}$.

2°. The function π is not uniquely defined by Eq.(1.6).

3°. If the form θ is exact, i.e., $\theta = d\varphi$, then the function φ is a solution of Eq.(1.6).

The domain of definition of quasicoordinates for non-closed forms has not yet been investigated. We know, nevertheless, that although the function π itself need not be defined throughout the phase space, its derivative along $\mathbf{X}$ can be continued to a smooth (analytic) function $\theta(\mathbf{X})$ on TV^n. This function is usually denoted by $\pi^{\cdot}$.

The introduction of π as a new coordinate generally requires explicit expression of π in terms of the old coordinates and vice versa.

The following example will illustrate a situation in which this complication does not arise. Let q_1 be a cyclic coordinate, i.e., it does not occur explicitly in the coefficients of the forms Ω and R in Eq.(1.4). In addition,

$$\theta = \Sigma\theta_i dq_i, \quad \theta_1 \neq 0, \quad R(\partial/\partial q_1) = 0$$

Expand Ω and R in terms of the forms $\theta, dq_2, \ldots, dq_n$ and collect all coefficients of θ in Ω in a form γ, so that $\Omega = \Omega_1 + \gamma \wedge \theta$. Under our assumptions we obtain $\gamma(\mathbf{X}) = 0$. This follows from a comparison of the coefficients of θ in Eq.(1.4).

Now put $\Lambda = \gamma \wedge (d\pi - \theta)$. Obviously, if π is a quasicoordinate corresponding to θ, then $\Lambda(\mathbf{X}, \cdot) = 0$.

Putting $\Omega_2 = \Omega + \Lambda$, we can write instead of Eq.(1.4) $\Omega_2(\mathbf{X}, \cdot) = (\Omega_1 + \gamma \wedge d\pi)(\mathbf{X}, \cdot) = R$, where the coordinate q_1 and differential dq_1 do not appear in Ω_2 and R, and we can treat π as a new coordinate. In Sect.2 we shall develop a procedure to introduce "arc length" as a coordinate describing the motion of a Chaplygin sleigh.

Remarks. 1. If $d\Omega_1 = 0$, $d\gamma = 0$, this procedure yields the Hamilton equations: $d\Omega_2 = 0$.

2. Since π is a cyclic coordinate in the new equations, its dependence on time does not affect the entire solution. The time-dependence of $\pi^{\cdot}$ is, however, essential.

2. Chaplygin's equations. If a non-holonomic system involves cyclic coordinates, as many as there are constraints, one can write the equations of the reduced system in a form similar to (1.4). Consider a conservative mechanical system with Hamiltonian $H^* = T^* - U^*$ and phase space TM^m, subject to k non-integrable, linear constraints which are homogeneous functions of the velocities:

$$f_s \equiv \sum_{i=1}^{m} a_i(\mathbf{q})\, q_i^{\cdot} = 0, \quad s = 1, \ldots, k \tag{2.1}$$

Suppose that neither the Hamiltonian H^* nor the functions a_i explicitly contain the coordinate q_s. Put $n = m - k$.

Theorem 1. The vector field of the reduced Chaplygin system may be defined on a certain space TV^n by the equation

$$\Omega(\mathbf{X}, \cdot) = -dH \tag{2.2}$$

where $\mathbf{X}$ is a special field. Here H is the restriction of the Hamiltonian H^* to the surface $\Sigma = \{\mathbf{f} = 0\}$, and Ω is a certain (not necessarily closed) 2-form.

We prove the theorem for the case in which the equations of the constraints can be written as

$$f_s \equiv q_s^{\cdot} - \sum_{i=k+1}^{m} a_{si} q_i^{\cdot} \tag{2.3}$$

Proof. The equations with Lagrange multipliers can be written

$$dd_v T^* (\mathbf{X}, \cdot) = -dH^* + \sum_{s=1}^{k} \lambda_s d_v f_s \tag{2.4}$$

and λ_s are found from the condition

$$L_{\mathbf{X}} f_s = df_s(\mathbf{X}) = 0 \tag{2.5}$$

It has been shown (2) that the field defined by Eqs. (2.4) is special on TM^m. Consequently,

$$d_v f_s(\mathbf{X})|_\Sigma = f_s|_\Sigma = 0 \tag{2.6}$$

In order to eliminate dependent velocities, we project the field $\mathbf{X}$ onto a suitable coordinate subspace of dimension $2m - k$. We then choose a function T that is identical with T^* on Σ and is independent of $q_s^{\cdot}$. Since T^* and T are identical on Σ, there exist functions h_s such that

$$T^* = T + \sum_{s=1}^{k} h_s f_s$$

and hence, substituting this expression into (2.4) and using (2.5), (2.6), we deduce that at points of Σ:

$$\left(dd_v T + \sum_{s=1}^{k} h_s dd_v f_s\right)(X, \cdot) = -dH + \sum_{s=1}^{k} (\lambda_s - L_{\mathbf{X}} h_s)\, d_v f_s$$

Since the coordinates q_s are cyclic, it follows that in this equation $d_v f_s$ is the only form involving dq_s. Therefore $\lambda_s - L_{\mathbf{X}} h_s = 0$. Substituting the derivatives $q_s^{\cdot}$ from (2.3) into h_s, we see that the resulting equation does not involve the variables q_s, $q_s^{\cdot}$ or their differentials.

The field $\mathbf{X}$ is defined up to terms proportional to $\partial/\partial q_s$ and $\partial/\partial q_s^{\cdot}$. We shall say that the field is reduced if the coefficients of $\partial/\partial q_s$ and $\partial/\partial q_s^{\cdot}$ vanish.

We finally obtain an equation for a special field on $TV^n = \{(q_{k+1}, \ldots, q_m, q_{k+1}^{\cdot}, \ldots, q_m^{\cdot})\}$:

$$\Omega(\mathbf{X}, \cdot) = \left(dd_v T + \sum_{s=1}^{k} h_s dd_v f_s\right)(\mathbf{X}, \cdot) = -dH \tag{2.7}$$

Note that the reduced equation is the restriction of the equation with multipliers to the 0-space of the forms $d_v\mathbf{f}$, $d\mathbf{f}$.

Corollary. If the system is natural, the form Ω is homogeneous with respect to the velocities.

Proof. Since T is quadratic, h_s is linear, so differentiation d does not affect the degree of homogeneity and d_v changes it by one.

Example. Consider the motion of a Chaplygin sleigh on a horizontal plane. Let x, y be the coordinates of the runner on the plane, φ the angle of rotation of the runner, a, b the coordinates of the centre of mass in a coordinate frame attached to the runner, and k is the radius of intertia. Then

$$2T = (x^{\cdot} - \varphi^{\cdot}(a \sin\varphi + b\cos\varphi))^2 + (y^{\cdot} + (a\cos\varphi - b\sin\varphi))^2 + k^2\varphi^{\cdot 2}$$

The equation of the constraint is

$$f \equiv x^{\cdot}\sin\varphi - y^{\cdot}\cos\varphi = 0$$

Changing to quasivelocities, we have

$$\varkappa^{\cdot} = x^{\cdot}\cos\varphi + y^{\cdot}\sin\varphi - b\varphi^{\cdot} \quad f = x^{\cdot}\sin\varphi - y^{\cdot}\cos\varphi$$

Then $2T = f^2 + \varkappa^{\cdot 2} - 2af\varphi^{\cdot} + r^2\varphi^{\cdot 2}$, $r^2 = a^2 + k^2$. The equation with multipliers is

$$\Omega(\mathbf{X}, \cdot) = (a\varphi^{\cdot} - f)\, df - \varkappa^{\cdot} d\varkappa^{\cdot} + (af - r^2\varphi^{\cdot})\, d\varphi^{\cdot} + \lambda d_v f$$

Using the equalities $dd_v f = d\varphi \wedge d_v\varkappa^{\cdot}$, $dd_v\varkappa^{\cdot} = -d\varphi \wedge d_v f$, we can write

$$\Omega = (df - \varkappa^{\cdot} d\varphi - ad\varphi^{\cdot}) \wedge d_v f + (d\varkappa^{\cdot} + (f - a\varphi^{\cdot})\, d\varphi) \wedge d_v\varkappa^{\cdot} + (r^2 d\varphi^{\cdot} - adf) \wedge d\varphi$$

Put $\gamma = d\varkappa^{\cdot} + (f - a\varphi^{\cdot})\, d\varphi$. Comparing the coefficients on $d_v\varkappa^{\cdot}$, we see that $\gamma(\mathbf{X}) = 0$.
Let $\varkappa$ be a function satisfying the equation $L_{\mathbf{X}}\varkappa = \varkappa^{\cdot}$ in its domain of definition. Then $\Lambda(\mathbf{X}, \cdot) = 0$, if $\Lambda = \gamma \wedge (d\varkappa - d_v\varkappa^{\cdot})$.
Consider the equation

$$(\Omega + \Lambda)(\mathbf{X}, \cdot) = -dT + \lambda d_v f$$

which yields the following reduced equation when one sets $f = 0$ and restricts all forms to the 0-space of df and $d_v f$ (**X** is a special field):

$$\Omega(\mathbf{X}, \cdot) = -dT$$
$$\Omega = d\varkappa^{\cdot} \wedge d\varkappa + r^2 d\varphi^{\cdot} \wedge d\varphi + a\varphi^{\cdot} d\varkappa \wedge d\varphi$$
$$2T = \varkappa^{\cdot 2} + r^2\varphi^{\cdot 2}, \quad \mathbf{X} = a_1\frac{\partial}{\partial\varkappa^{\cdot}} + a_2\frac{\partial}{\partial\varphi^{\cdot}} + \varkappa^{\cdot}\frac{\partial}{\partial\varkappa} + \varphi^{\cdot}\frac{\partial}{\partial\varphi}$$

The transformation to quasicoordinates and the conjugate quasivelocities does not induce any diffeomorphism of the configuration space V^n. Consequently, if a transformation of this type results in a phase space with the structure of a tangent bundle, this structure differs from the previous structure. The old coordinates and velocities have been mixed.

Below we shall consider equations of the type (2.2) with a form Ω, homogeneous of degree one, satisfying the non-singularity condition

$$\Omega^n = g\omega, \quad g \neq 0 \quad (\omega = dq_1 \wedge \ldots \wedge dq_n \wedge dq_1^{\cdot} \wedge \ldots \wedge dq_n^{\cdot})$$

where ω is the form of the volume.
It can be shown that Ω can be expressed as

$$\Omega = d\alpha + \sigma, \quad \sigma = \Sigma\sigma_{ij}^{k}(\mathbf{q})\, q_k^{\cdot} dq_i \wedge dq_j \tag{2.8}$$

where α is some 1-form.

3. The Chaplygin reducing factor.

A change of time variable corresponds to re-parametrization of the integral curve or, what is the same, multiplication of the velocity vector by a certain function. The integral curves themselves remain unchanged.

Definition 3. A vector field **X** defined as in (2.2) has a Chaplygin reducing factor if there exists a function $N > 0$ such that the field $\mathbf{X}/N$ is Hamiltonian with the same Hamiltonian, i.e., for some closed differential 2-form Ω_1,

$$\Omega_1(\mathbf{X}/N, \cdot) = -dH, \quad d\Omega_1 = 0 \tag{3.1}$$

Conditions for the existence of a reducing factor where discussed in detail in /3/. The basic conditions are as follows.

Proposition 1. A reducing factor for a field X defined by (2.2) exists if and only if there exist a function $N > 0$ and a 2-form Λ such that

$$d\Omega_1 = 0, \quad \Omega_1 = N\Omega + \Lambda, \quad \Lambda(\mathbf{X}, \cdot) = 0$$

This is simply a rephrasing of the definition. The following condition is less general but more constructive.

Proposition 2. If there exists a function P on TV^n such that

$$dP \wedge \alpha = \sigma \tag{3.2}$$

then the field **X** defined by (2.2) and (2.8) has the reducing factor $N = \exp P$.

Proof. Put $N = \exp P$. Then by (3.2) we have $dN \wedge \alpha = N\sigma$. Hence

$$d(N\Omega) = d(Nd\alpha + N\sigma) = d(Nd\alpha + dN \wedge \alpha) = 0$$

and to satisfy condition (3.1) we need only put $\Omega_1 = N\Omega$.

Corollary. If P is twice differentiable in the neighbourhood of A, then there exists a reducing factor $N_0 = N_0(\mathbf{q})$.

Proof. By Hadamard's Lemma,

$$P = P|_{\mathbf{q}^\cdot=0} + \sum_{i=1}^{n} q_i^\cdot P_i(\mathbf{q}, \mathbf{q}^\cdot) = P_0 + \sum_{i=1}^{n} q_i^\cdot P_i$$

where P is differentiable with respect to the velocities. Eq.(3.2) may be written

$$(dP_0 \wedge \alpha - \sigma) + d(\Sigma q_i^\cdot P_i) \wedge \alpha = 0 \tag{3.3}$$

Since the forms α and σ are homogeneous of degree one, it follows that (3.3) is a set of equations of the form

$$\Sigma q_i^\cdot a_i(\mathbf{q}) + \Sigma q_i^\cdot q_j^\cdot b_{ij}(\mathbf{q}, \mathbf{q}^\cdot) = 0 \tag{3.4}$$

where a_i and b_{ij} are certain functions, and the first term of (3.4) corresponds to the first term of (3.3).

Differentiating (3.4) with respect to $\mathbf{q}^\cdot$ and putting $\mathbf{q}^\cdot = 0$, we see that $a_i = 0$. Consequently, the first term of (3.3) also vanishes, i.e.,

$$dP_0 \wedge \alpha = \sigma, \quad N_0 = \exp P_0$$

The following fact is also worthy of note.

Theorem 2. If the field $\mathbf{X}$ defined by Eq.(2.2) has a reducing factor and the form $N\Omega$ is closed, then there exists an integral invariant with density $\mu = gN^{n-1}$, where g is defined by the equality $\Omega^n = g\omega$, and ω is the form of the volume.

Proof. By assumption, $N\Omega(\mathbf{X}/N, \cdot) = -dH$ and $d(N\Omega) = 0$. Let $\mathbf{X}/N = \mathbf{Y}$, then $L_{\mathbf{Y}}(N\Omega) = -ddH = 0$ and

$$0 = n(L_{\mathbf{Y}} N\Omega) \wedge (N\Omega)^{n-1} = L_{\mathbf{Y}}(N\Omega)^n = L_{\mathbf{X}} N^{n-1}\Omega^n = L_{\mathbf{X}}(\mu\omega)$$

The assumption that $d\alpha$ and σ are homogeneous yields a number of useful conclusions.

4. Invariant measure. The equation defining the density of the integral invariant $\mu\omega$ may be written differently, e.g.,

$$L_{\mathbf{X}}(\mu\omega) = 0 \tag{4.1}$$

It is convenient to define $W = \ln(\mu/g)$. Then

$$L_{\mathbf{X}} W = \varphi \tag{4.2}$$

where φ is some function.

Lemma 1. Let $H = T - U$, let $d\alpha$ and σ be homogeneous of degree one and the field $\mathbf{X}$ defined by (2.2). Then φ is linear in the velocities and independent of the potential and potential forces.

Proof. Using the equality $g\omega = \Omega^n$ and performing some reduction, we obtain the equation

$$\omega L_{\mathbf{X}} W = -d\sigma(\mathbf{X}, \cdot) \wedge \Omega^{n-1} n/g \tag{4.3}$$

On either side of this equation we have $2n$-forms, i.e, products of the volume form by a coefficient. The coefficient on the right is φ.

The form ω involves the product of n differentials $dq_i^\cdot$. The factor Ω cannot contribute more than one to the product. Each term of the form $d\sigma(\mathbf{X}, \cdot)$ also involves at most one velocity differential. If

$$\sigma = \Sigma \sigma_{ij}^k q_k^\cdot dq_i \wedge dq_j$$

these terms are

$$\sum_{i<j} \sigma_{ij}^k dq_k^\cdot \wedge (q_i^\cdot dq_j - q_j^\cdot dq_i)$$

i.e., the coefficients of the form $d\sigma(\mathbf{X}, \cdot)$ in these terms are linear in the velocities.

The form Ω is homogeneous of degree one, and therefore the coefficients of $dq_i^\cdot$ in it do not involve the velocities. To complete the proof, we note that the expression on the right of (4.3) involves only the coefficients of the forms $d\alpha$ and σ, and not the components of the field $\mathbf{X}$, which depend on the potential.

Theorem 3. If the special field X on TV^n defined by (2.2) and (2.8) has an integral invariant with density $\mu = \mu(\mathbf{q})$ for $H = T$, i.e., for $U = 0$, then the same function $\mu(\mathbf{q})$ is also the density of the integral invariant for the field obtained when dU is added on the right of (2.2), whatever the potential $U = U(\mathbf{q})$.

Proof. By the lemma, nothing on the right-hand side of (4.2) is changed by the addition of dU. But the left-hand side also remains unchanged, because if $W = W(\mathbf{q})$, then

$$L_X W = \Sigma q_i^\cdot \partial W/\partial q_i$$

i.e., the field components, which depend on the potential, do not act on functions of $\mathbf{q}$.

Thus the density is determined by the same equation as before.

The homogeneity property enables us to draw conclusions as to the local organization of the invariant measure in potential-free systems.

Theorem 4. If the special field $\mathbf{X}$ on TV^n defined by (2.2) and (2.8) has an integral invariant when $U = 0$, with density $\mu = \mu(\mathbf{q}, \mathbf{q}^{\cdot})$ twice differentiable with respect to the velocities in the neighbourhood of the set A, and the kinetic energy is a quadratic function of the velocities, then

$$\mu(\mathbf{q}, \mathbf{q}^{\cdot}) = \mu_0(\mathbf{q}) F(\mathbf{q}, \mathbf{q}^{\cdot})$$

where μ_0 is the density of the integral and F is a first integral.

Proof. Since T is quadratic and Ω is homogeneous, it follows that $\mathbf{X}$ itself is homogeneous of degree 1. The rest of the proof is analogous to that of the corollary to Proposition 1 and is based on the expansion

$$W = \ln \mu = W|_{q^{\cdot}=0} + \Sigma q_i^{\cdot} W_i = W_0 + \Sigma q_i^{\cdot} W_i = \ln \mu_0 + \ln F$$

Example. Consider the motion of a non-holonomic Chaplygin sphere on a horizontal plane. In $R^6 = R_\omega^3 \times R_\gamma^3$ this motion is described by the equation

$$\begin{aligned}
&\mathbf{k}^{\cdot} + \boldsymbol{\omega} \times \mathbf{k} = 0, \quad \gamma^{\cdot} + \boldsymbol{\omega} \times \gamma = 0 \\
&k = I\omega + ma^2\gamma \times (\omega \times \gamma), \quad I = \operatorname{diag}(I_1, I_2, I_3) \\
&\omega = \operatorname{col}(p, q, r), \quad \gamma = \operatorname{col}(\gamma_1, \gamma_2, \gamma_3)
\end{aligned} \tag{4.4}$$

These equations have an integral $\langle \gamma, \gamma \rangle = \rho^2$ ($\langle , \rangle$ denotes convolution). Define spherical coordinates on the surface $\rho = 1$:

$$\begin{aligned}
&\gamma_1 = \rho s_2 s_3, \quad \gamma_2 = \rho s_2 c_3, \quad \gamma_3 = \rho c_2; \quad J = -\rho^2 s_2, \\
&J_\rho = \det \| \partial(\gamma_1, \gamma_2, \gamma_3)/\partial(\rho, \theta, \varphi) \| \\
&s_2 = \sin\theta, \quad c_2 = \cos\theta, \quad s = \sin\varphi, \quad c_3 = \cos\varphi
\end{aligned} \tag{4.5}$$

Embedding this surface in a space with coordinates $(\psi, \theta, \varphi, p, q, r)$ (we have added ψ), we let $\mathbf{X}_s$ denote the field corresponding to equations (4.4):

$$\mathbf{X}_s = a_1 \frac{\partial}{\partial p} + a_2 \frac{\partial}{\partial q} + a_3 \frac{\partial}{\partial r} + (pc_3 - qs_3)\frac{\partial}{\partial \theta} + \left(r - p\frac{c_2 s_3}{s_2} - q\frac{c_2 c_3}{s_2}\right)\frac{\partial}{\partial \varphi} \tag{4.6}$$

where a_1, a_2, a_3 are functions of $\gamma(\theta, \varphi)$.

By Theorem 1, the equations of motion of the sphere can also be obtained in the form of (2.2), with $TV = \{(\psi^{\cdot}, \theta^{\cdot}, \varphi^{\cdot}, \psi, \theta, \varphi)\}$. Transforming to angular velocities,

$$p = \psi^{\cdot} s_2 s_3 + \theta^{\cdot} c_3, \quad q = \psi^{\cdot} s_2 c_3 - \theta^{\cdot} s_3, \quad r = \psi^{\cdot} c_2 + \varphi^{\cdot} \tag{4.7}$$

we let $\mathbf{X}_c$ denote the corresponding special field.

Proposition 3. $\mathbf{X}_c = \mathbf{X}_s + \mathbf{X}_\psi$, where $\mathbf{X}_\psi = \psi^{\cdot}\partial/\partial\psi$.

Proof. Define column vectors

$$d_v\omega = \begin{Vmatrix} d_v p \\ d_v q \\ d_v r \end{Vmatrix} = \begin{Vmatrix} d\psi s_2 s_3 + d\theta c_3 \\ d\psi s_2 c_3 - d\theta s_3 \\ d\psi c_2 + d\varphi \end{Vmatrix}, \quad dd_v\omega = \begin{Vmatrix} d_v r \wedge d_v q \\ d_v p \wedge d_v r \\ d_v q \wedge d_v p \end{Vmatrix}$$

and a matrix

$$B = \begin{Vmatrix} \gamma_2^2 + \gamma_3^2 & -\gamma_1\gamma_2 & -\gamma_1\gamma_3 \\ -\gamma_1\gamma_2 & \gamma_1^2 + \gamma_3^2 & -\gamma_2\gamma_3 \\ -\gamma_1\gamma_3 & -\gamma_2\gamma_3 & \gamma_1^2 + \gamma_2^2 \end{Vmatrix}$$

The form Ω can be written out explicitly:

$$\begin{aligned}
&\Omega = d\alpha + \sigma, \quad \sigma = -\langle ma^2 B\omega, dd_v\omega \rangle \\
&d\alpha = d\langle (I + ma^2B)\,\omega, d_v\omega \rangle = \langle (I + ma^2B)\, d\omega, d_v\omega \rangle + \langle I\omega, dd_v\omega \rangle \\
&g = -3! \cdot s_2 \det \| I + ma^2 B \|
\end{aligned}$$

Multiplication, wherever necessary, is understood as outer multiplication of forms; $\gamma_1, \gamma_2, \gamma_3$ are the functions of the Euler angles θ, φ defined by (4.5).

It can be verified directly that

$$\mathbf{X}_c = a_1 \frac{\partial}{\partial p} + a_2 \frac{\partial}{\partial q} + a_3 \frac{\partial}{\partial r} + \psi^{\cdot} \frac{\partial}{\partial \psi} + \theta^{\cdot} \frac{\partial}{\partial \theta} + \varphi^{\cdot} \frac{\partial}{\partial \varphi} = \mathbf{X}_s + \mathbf{X}_\psi$$

where a_1, a_2, a_3 are the same as in (4.6).

Put $J = J_\rho|_{\rho=1} = -s_2$.

Lemma 2. The vector field $\mathbf{X}_c$ has the invariant measure $\mu J\omega_1$, where

$$\omega_1 = d\psi \wedge d\theta \wedge d\varphi \wedge dp \wedge dq \wedge dr$$
$$\mu = [(ma^2)^{-1} - \langle\gamma, (I + ma^2E)^{-1}\gamma\rangle]^{-1/2}$$

Proof. This was proved for $\mathbf{X}_5$ in /5/, and $\mathbf{L}_{\mathbf{X}_\psi}(\mu J\omega_1) = 0$ follows from the fact that ψ does not occur in any of the functions involved.

We now observe that although the functions $\mu_1 = \mu J$ and g may vanish, owing to singularities in the system of coordinates (Euler angles), the quotient μ_1/g always remains positive, and application of Theorem 3 yields the following simple proposition.

Proposition 4. For any function $U = U(\gamma)$, the differential equations

$$\mathbf{k}^\cdot + \boldsymbol{\omega} \times \mathbf{k} = \gamma \times U_\gamma', \quad \gamma^\cdot + \boldsymbol{\omega} \times \gamma = 0 \tag{4.8}$$

in $R^6 = R_\omega^3 \times R_\gamma^3$ have an invariant measure $\mu\omega_2$ (ω_2 is the form of the volume, $\omega_2 = d\gamma_1 \wedge d\gamma_2 \wedge d\gamma_3 \wedge dp \wedge dq \wedge dr$).

It can be verified that (4.8) are the equations for a Chaplygin sphere with arbitrary potential.

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